Multisoliton solutions to the lattice Boussinesq equation

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Abstract

The lattice Boussinesq equation (BSQ) is a three-component difference-difference equation defined on an elementary square of the 2D lattice, having 3D consistency. We write the equations in the Hirota bilinear form and construct their multisoliton solutions in terms of Casoratians, following the methodology in our previous papers. In the construction it turns out that instead of the usual discretization of the exponential as $[(a+k)/(a-k)]^n$ we need two different terms $[(a-\omega k)/(a-k)]^n$ and $[(a-\omega^2 k)/(a-k)]^n$, where ω is a cubic root of unity $\neq 1$.

1 Introduction

Among the integrable 2D difference equations one important class consists of those equations that are defined on an elementary quadrilateral and are multidimensional consistent. Here multidimensional consistency means that one can add a third dimension and extend the definition in a natural way from a quadrilateral to a cube, and that on this cube the maps are consistent.

Most of the maps in this class are included in the ABS classification [1], (which is complete within the assumptions of symmetry and the tetrahedron property). These lattice maps have one component for each lattice site, but there are also 3D-consistent *multicomponent* maps related to the Boussinesq (BSQ) equation [2, 3, 4, 5].

In this paper we will derive multisoliton solutions to the lattice BSQ equation defined on the elementary square by the equations [5]

$$B1 \equiv \widetilde{w} - u\widetilde{u} + v = 0, \tag{1.1a}$$

$$B2 \equiv \widehat{w} - u\widehat{u} + v = 0, \tag{1.1b}$$

$$B3 \equiv w - u\widehat{\widetilde{u}} + \widehat{\widetilde{v}} - \frac{p-q}{\widehat{u}-\widetilde{u}} = 0, \tag{1.1c}$$

where we have used the standard shorthand notation, e.g., $\widetilde{u} = u_{n+1,m}$, $\widehat{v} = v_{n,m+1}$, and where p and q are parameters in the n and m directions, respectively. From equations (1.1a),(1.1b)

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one finds, after taking the hat and tilde shifts, respectively

$$\widehat{\widetilde{u}} = \frac{\widetilde{v} - \widehat{v}}{\widetilde{u} - \widehat{u}}, \tag{1.2a}$$

$$\widehat{\widetilde{w}} = \frac{\widetilde{u}\widehat{v} - \widehat{u}\widetilde{v}}{\widetilde{u} - \widehat{u}}. \tag{1.2b}$$

Equations (1.1),(1.2) are consistent on the elementary square.

Three-dimensional consistency means that we can add a third direction, with parameter r. The new equations will be

$$\overline{w} - u\overline{u} + v = 0, \tag{1.3a}$$

$$w - u\widehat{\overline{u}} + \widehat{\overline{v}} - \frac{r-q}{\widehat{u} - \overline{u}} = 0, \tag{1.3b}$$

$$w - u\widetilde{\overline{u}} + \widetilde{\overline{v}} - \frac{r-p}{\widetilde{u} - \overline{u}} = 0, \tag{1.3c}$$

where we have denoted the shift in the third direction by a bar. To this we should add the bar-tilde and bar-hat versions of (1.2), which can be derived from (1.3a).

Consider now the cube of Figure 1, where F stands for the three components (u, v, w). The

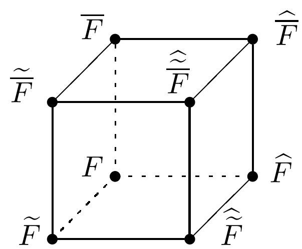

Figure 1: The multi-dimensional consistency cube.

initial values are given at F, $\widetilde{F}$, $\widehat{F}$, $\overline{F}$, but due to the linear equations (1.1a),(1.1b),(1.3a) we only need to give (u, v, w), $(\widetilde{u}, \widetilde{v})$, $(\widehat{u}, \widehat{v})$, $(\overline{u}, \overline{v})$. Given these values we can compute $(\widehat{\widetilde{u}}, \widehat{\widetilde{v}}, \widehat{\widetilde{w}})$, $(\widehat{\overline{u}}, \widehat{\overline{v}}, \widehat{\overline{w}})$ $(\widetilde{\overline{u}}, \widetilde{\overline{v}}, \widetilde{\overline{w}})$ using (1.2) and its tilde-bar and hat-bar versions, along with (1.1c),(1.3b),(1.3c). After this there are three different ways to compute the remaining $(\widehat{\widetilde{\overline{u}}}, \widehat{\widetilde{\overline{v}}}, \widehat{\widetilde{\overline{w}}})$ but they all give the same values and the system is therefore 3D-consistent.

In the following construction we make active use the 3D-view with different interpretations for the bar shift. In Section 2 we construct first the background solution, then the one-soliton solution, and then in Section 3 we derive the Hirota bilinear form and propose and prove the formulae for N-soliton solutions. The procedure is in principle similar to the one used in [6, 7, 8], but the multicomponent nature induces some new features.

2 The background and one-soliton solutions

The fixed-point idea proposed in [6] for the construction of the background solution (0SS) is that the background or "seed" solution should be a fixed point with respect to a shift in the

third direction, i.e., $\overline{u} = u, \overline{v} = v, \overline{w} = w$. The relevant equations (the equations on the sides of the cube) are then obtained from (1.3) as

$$\left\{ \begin{array}{l} w = u^2 - v, \\ w = u\widetilde{u} - \widetilde{v} + \frac{p-r}{u-\widetilde{u}}, \\ w = u\widehat{u} - \widehat{v} + \frac{r-q}{\widehat{u}-u}, \end{array} \right. \tag{2.1}$$

where r is the parameter in the bar-direction and now plays the role of a parameter of the background solution. The equations are easy to solve and we find the 0SS

$$u_0 = an + bm + c_1, \tag{2.2a}$$

$$v_0 = \tfrac{1}{2}u_0^2 + \tfrac{1}{2}(a^2 n + b^2 m + c_2) + c_3, \tag{2.2b}$$

$$w_0 = \tfrac{1}{2}u_0^2 - \tfrac{1}{2}(a^2 n + b^2 m + c_2) - c_3, \tag{2.2c}$$

where a, b are related to p, q by

$$a^3 = r - p, \quad b^3 = r - q, \tag{2.3}$$

and c_1, c_2, c_3 are arbitrary constants.

The one-soliton solution (1SS) is constructed with the same idea of using the 3D cube, but now u, v, w correspond to the background solution and $\overline{u}, \overline{v}, \overline{w}$ to the 1SS. In more detail, we have the side equations

$$\overline{w} = u\overline{u} - v, \tag{2.4a}$$

together with

$$\left\{ \begin{array}{l} \widetilde{\overline{u}} = \frac{\widetilde{v}-\overline{v}}{\widetilde{u}-\overline{u}}, \\ \widetilde{\overline{v}} = u\widetilde{\overline{u}} - w + \frac{k^3-a^3}{\overline{u}-\widetilde{u}}, \end{array} \right. \qquad \left\{ \begin{array}{l} \widehat{\overline{u}} = \frac{\overline{v}-\widehat{v}}{\overline{u}-\widehat{u}}, \\ \widehat{\overline{v}} = u\widehat{\overline{u}} - w + \frac{b^3-k^3}{\widehat{u}-\overline{u}}. \end{array} \right. \tag{2.4b}$$

These act as a Bäcklund transformation (BT) with k as the BT parameter related to the bar direction.

In order to solve equations (2.4) we take $(u, v, w) = (u_0, v_0, w_0)$ where the background solution is defined in (2.2). The 1SS is written in the form

$$(\overline{u}, \overline{v}, \overline{w}) = (\overline{u}_0 + x, \overline{v}_0 + y, \overline{w}_0 + z), \tag{2.5}$$

where $(\overline{u}_0, \overline{v}_0, \overline{w}_0)$ is the bar-shifted (u_0, v_0, w_0), i.e.,

$$\overline{u}_0 = an + bm + k + c_1, \tag{2.6a}$$

$$\overline{v}_0 = \tfrac{1}{2}\overline{u}_0^2 + \tfrac{1}{2}(a^2 n + b^2 m + k^2 + c_2) + c_3, \tag{2.6b}$$

$$\overline{w}_0 = \tfrac{1}{2}\overline{u}_0^2 - \tfrac{1}{2}(a^2 n + b^2 m + k^2 + c_2) - c_3. \tag{2.6c}$$

With these definitions we find from (2.4a) that $z = u_0 x$. Thus we only need to solve for x, y, for which we have from (2.4b)

$$\left\{ \begin{array}{lcl} \widetilde{x} & = & \frac{-\widetilde{\overline{u}}_0\, x + y}{x - a + k}, \\ \widehat{x} & = & \frac{-\widehat{\overline{u}}_0\, x + y}{x - b + k}, \end{array} \right. \qquad \left\{ \begin{array}{lcl} \widetilde{y} & = & \frac{-(\widetilde{\overline{v}}_0 + w_0)\, x + u_0\, y}{x - a + k}, \\ \widehat{y} & = & \frac{-(\widehat{\overline{v}}_0 + w_0)\, x + u_0\, y}{x - b + k}, \end{array} \right. \tag{2.7}$$

This system can be linearized by taking $(x, y) = (\frac{G}{F}, \frac{H}{F})$, the result is

$$\widetilde{\Psi} = N\Psi, \quad \widehat{\Psi} = M\Psi, \tag{2.8a}$$

where

$$\Psi = \begin{pmatrix} G \\ H \\ F \end{pmatrix}, \quad N = \begin{pmatrix} \widetilde{\widetilde{u}}_0 & -1 & 0 \\ \widetilde{\widetilde{v}}_0 + w_0 & -u_0 & 0 \\ -1 & 0 & a-k \end{pmatrix}, \quad M = \begin{pmatrix} \widehat{\widehat{u}}_0 & -1 & 0 \\ \widehat{\widehat{v}}_0 + w_0 & -u_0 & 0 \\ -1 & 0 & b-k \end{pmatrix}. \tag{2.8b}$$

The matrices N, M satisfy the integrability condition $\widehat{N}M = \widetilde{M}N$.

In order to construct the solutions it is useful to note that if we define

$$Q(n,m) = \begin{pmatrix} u_0(n,m) - \omega k & -1 & 0 \\ u_0(n,m) - \omega^2 k & -1 & 0 \\ (-u_0(n,m) + k)/(3k^2) & 1/(3k^2) & 1 \end{pmatrix}, \tag{2.9}$$

then

$$N = Q(n+1,m)^{-1}\, D(a)\, Q(n,m), \quad \text{where} \quad D(a) = \begin{pmatrix} a - \omega k & 0 & 0 \\ 0 & a - \omega^2 k & 0 \\ 0 & 0 & a-k \end{pmatrix}, \tag{2.10}$$

and similarly $M = Q(n, m+1)^{-1}\, D(b)\, Q(n,m)$. Here ω is a cubic root of unity, $\omega \neq 1$, i.e, $\omega^2 + \omega + 1 = 0$. Using these we can straightforwardly construct the solution as

$$\Psi(n,m) = Q(n,m)^{-1} D(a)^n D(b)^m Q(0,0)\Psi(0,0), \tag{2.11}$$

from which we find

$$G = k(\omega - 1)\left[\rho_1^0(a-\omega k)^n(b-\omega k)^m - \omega^2\rho_2^0(a-\omega^2 k)^n(b-\omega^2 k)^m\right], \tag{2.12a}$$

$$H = u_0\, g + k^2(\omega-1)\left[-\omega^2\rho_1^0(a-\omega k)^n(b-\omega k)^m + \rho_2^0(a-\omega^2 k)^n(b-\omega^2 k)^m\right], \tag{2.12b}$$

$$F = \rho_0^0(a-k)^n(b-k)^m + \rho_1^0(a-\omega k)^n(b-\omega k)^m + \rho_2^0(a-\omega^2 k)^n(b-\omega^2 k)^m, \tag{2.12c}$$

where we have introduced new constants ρ_ν^0 in place of G_{00}, H_{00}, F_{00}. Using (2.5),(2.6) we can recover the 1SS as

$$u^{1SS} = u_0 + k\frac{1 + \omega\rho_1 + \omega^2\rho_2}{1 + \rho_1 + \rho_2}, \tag{2.13a}$$

$$v^{1SS} = v_0 + u_0\, k\frac{1 + \omega\rho_1 + \omega^2\rho_2}{1 + \rho_1 + \rho_2} + k^2\,\frac{1 + \omega^2\rho_1 + \omega\rho_2}{1 + \rho_1 + \rho_2}, \tag{2.13b}$$

$$w^{1SS} = w_0 + u_0\, k\frac{1 + \omega\rho_1 + \omega^2\rho_2}{1 + \rho_1 + \rho_2}. \tag{2.13c}$$

Here u_0, v_0, w_0 were defined in (2.2) and

$$\rho_\nu(n,m;k) = \frac{(a-\omega^\nu k)^n}{(a-k)^n}\frac{(b-\omega^\nu k)^m}{(b-k)^m}\frac{\rho_\nu^0}{\rho_0^0}, \quad \nu = 1, 2, \tag{2.14}$$

where k is the soliton parameter.

3 Bilinearization, Casoratians and N-soliton solutions

3.1 The main result

The 1SS (2.13) is not quite sufficient for guessing the general structure for NSS, but after considering also 2SS (in the case $\rho_2^0 = 0$) we arrived to a solution in terms of Casoratians, constructed as follows: Given the multi-indexed function

$$\psi_j(n,m,l) = \sum_{s=1}^{3} \varrho_{j,s}^{(0)} (\delta - \omega^s k_j)^l (a - \omega^s k_j)^n (b - \omega^s k_j)^m, \tag{3.1}$$

we define the column vector

$$\psi(n,m,l) = (\psi_1(n,m,l), \psi_2(n,m,l), \cdots, \psi_N(n,m,l))^T, \tag{3.2}$$

and then the generic $N \times N$ Casorati matrix by combining columns with different shifts l_i. The generic Casoratian is then the determinant

$$C_{n,m}(\psi; \{l_i\}) = |\psi(n,m,l_1), \psi(n,m,l_2), \cdots, \psi(n,m,l_N)|. \tag{3.3}$$

To describe such Casoratians we use the shorthand notation [9] in which only the shifts are given. Furthermore for consecutive sequences we use $\widehat{M} \equiv 0, 1, \ldots, M$ (this cannot be confused with the use of hat for shifts).

Proposition 1. *Multisoliton solutions to* (1.1) *are given by*

$$u = u_0 - \frac{g}{f}, \quad v = v_0 - u_0 \frac{g}{f} + \frac{h}{f}, \quad w = w_0 - u_0 \frac{g}{f} + \frac{s}{f}, \tag{3.4}$$

where u_0, v_0, w_0 are given in (2.2) *and the functions f, g, h, s are given in terms of Casoratians**

$$f = |\widehat{N-1}|,\ g = |\widehat{N-2}, N|,\ h = |\widehat{N-2}, N+1|,\ s = |\widehat{N-3}, N-1, N|, \tag{3.5}$$

composed of ψ given in (3.1) *with $\delta = 0$.*

Here the size on the matrix N indicates the number of solitons and the set $\{k_i\}_{i=1}^N$ provides the "velocity" parameters of the solitons, while the parameters $\varrho_{j,s}^{(0)}$ are related to the locations of the solitons (by gauge invariance only their ratio is significant).

In order to prove the above Proposition, we note that using (3.4) as a dependent variable transformation we can bilinearize (1.1) as

$$\mathcal{B}_1 = \widetilde{f}(h + ag) - \widetilde{g}(g + af) + f\widetilde{s} = 0, \tag{3.6a}$$

$$\mathcal{B}_2 = \widehat{f}(h + bg) - \widehat{g}(g + bf) + f\widehat{s} = 0, \tag{3.6b}$$

$$\mathcal{B}_3 = \widetilde{f}\widehat{g} - \widehat{f}\widetilde{g} - (a-b)(\widetilde{f}\widehat{f} - f\widehat{\widetilde{f}}) = 0, \tag{3.6c}$$

$$\mathcal{B}_4 = (a^2 + ab + b^2)(f\widehat{\widetilde{f}} - \widetilde{f}\widehat{f}) + (a+b)(f\widehat{\widetilde{g}} - \widehat{\widetilde{f}}g) + \widehat{\widetilde{f}}s + f\widehat{\widetilde{h}} - g\widehat{\widetilde{g}} = 0. \tag{3.6d}$$

* If $N = 1$ then $f = |0|, g = |1|, h = |2|, s = 0$; and if $N = 2$ then $f = |0,1|, g = |0,2|, h = |0,3|, s = |1,2|$.

In fact, the lattice BSQ equation (1.1) can be written as

$$B1 = \frac{\mathcal{B}_1}{f\widetilde{f}} \quad B2 = \frac{\mathcal{B}_2}{f\widehat{f}} \quad (\widehat{u}-\widetilde{u})B3 = \frac{\mathcal{B}_3\mathcal{B}_4 - (a-b)f\widehat{\widetilde{f}}\mathcal{B}_4 + (a^2+ab+b^2)\widetilde{f}\widehat{f}\mathcal{B}_3}{f\widetilde{f}\widehat{f}\widehat{\widetilde{f}}}. \tag{3.7}$$

Thus we need to prove that the Casoratians (3.5) solve the bilinear equations (3.6). This is given in Appendix A for the generic values of δ, although only $\delta = 0$ is used for the lattice BSQ. The Casoratian proof suggested the $\delta \neq 0$ generalization along with some others, they are discussed next.

3.2 The role of δ

The solution to the lattice BSQ are obtained with $\delta = 0$ in the matrix entry (3.1) but the generalization $\delta \neq 0$ is natural and we may ask about its meaning. It is important to note that the proof given in the Appendix can be carried out using only the following assumptions on the column vectors

$$\begin{aligned}
(a-\delta)\underset{\sim}{\psi} &= \psi - \underset{\sim}{\overline{\psi}}, && (3.8a)\\
(b-\delta)\underset{\wedge}{\psi} &= \psi - \underset{\wedge}{\overline{\psi}}, && (3.8b)\\
\gamma\psi &= \overline{\overline{\overline{\psi}}} - 3\delta\overline{\overline{\psi}} + 3\delta^2\overline{\psi}, && (3.8c)
\end{aligned}$$

where γ is a diagonal matrix and the bar-shift of ψ is defined by $\overline{\psi}(n,m,l) = \psi(n,m,l+1)$. From these assumptions one can derive the bilinear equations

$$\mathcal{B}_1^\delta = \widetilde{f}[h+(a-\delta)g] - \widetilde{g}[g+(a-\delta)f] + f\widetilde{s} = 0, \tag{3.9a}$$

$$\mathcal{B}_2^\delta = \widehat{f}[h+(b-\delta)g] - \widehat{g}[g+(b-\delta)f] + f\widehat{s} = 0, \tag{3.9b}$$

$$\mathcal{B}_3^\delta = \widetilde{f}\widehat{g} - \widehat{f}\widetilde{g} + (a-b)(\widetilde{f}\widehat{f} - f\widehat{\widetilde{f}}) = 0, \tag{3.9c}$$

$$\mathcal{B}_4^\delta = (a^2+ab+b^2)(f\widehat{\widetilde{f}} - \widetilde{f}\widehat{f}) + (a+b+\delta)(\widehat{\widetilde{f}}g - f\widehat{\widetilde{g}}) + \widehat{\widetilde{f}}s + f\widehat{\widetilde{h}} - g\widehat{\widetilde{g}} = 0. \tag{3.9d}$$

We can now reverse the dependent variable transformation (3.4) and construct from (3.9) a generalized lattice BSQ equation

$$\widetilde{w} = u\widetilde{u} - v + \delta(\widetilde{u} - u - a), \tag{3.10a}$$

$$\widehat{w} = u\widehat{u} - v + \delta(\widehat{u} - u - b), \tag{3.10b}$$

$$w = u\widehat{\widetilde{u}} - \widehat{\widetilde{v}} + \frac{-a^3+b^3}{\widehat{u}-\widetilde{u}} - \delta(\widehat{\widetilde{u}} - u - a - b). \tag{3.10c}$$

Obviously, if we take $\delta = 0$, the above equations reduce to those of lattice BSQ.

Now considering the Casoratian forms of f, g, h, s one can easily show that

$$\begin{aligned}
f(\delta) &= f(0),\\
g(\delta) &= g(0) + N\,\delta\, f(0),\\
h(\delta) &= h(0) + (N+1)\,\delta\, g(0) + N(N+1)/2\,\delta^2\, f(0),\\
s(\delta) &= s(0) + (N-1)\,\delta\, g(0) + N(N-1)/2\,\delta^2\, f(0),
\end{aligned}$$

where N is the dimension of the matrix. From this result, which was derived from a particular form of the solution, we are led to the following: If we denote the δ-dependent functions in (3.10) by u', v', w' then the transformation

$$u' = u, \quad v' = v - \delta\,(u - u_0), \quad w' = w + \delta\,(u - u_0), \tag{3.11}$$

converts (3.10) into (1.1). Thus the effect of introducing δ in (3.1) can be undone by the "gauge" transformation (3.11).

3.3 Toeplitz generalization

Another generalization in the construction of the solution is obtained if we replace (3.8c) with

$$\Gamma\psi = \overline{\overline{\overline{\psi}}} - 3\delta\overline{\overline{\psi}} + 3\delta^2\overline{\psi}, \tag{3.12}$$

where Γ is some $N \times N$ matrix. Under this generalization proof of the bilinear equations (3.9) proceeds as before, except for some details described in Appendix A.3.

First note that since the solutions are given in terms of determinants any matrix similar to Γ yields same solution as Γ. Thus it is sufficient to consider only different canonical forms of Γ.

If Γ is a diagonal matrix

$$\Gamma = \mathrm{Diag}(\gamma_1, \gamma_2, \cdots, \gamma_N) \tag{3.13a}$$

where

$$\gamma_j = \delta^3 - k_j^3, \tag{3.13b}$$

with distinct $\{k_j\}$, then the entries in the Casoratian can be as in (3.1). Since $\omega^2 = \omega^*$ (where $*$ stands for complex conjugate), the condition for a real solution (coming from real ψ_j) is

$$\varrho_{j,2}^{(0)} = \varrho_{j,1}^{(0)*}, \quad \text{with} \quad k_j, \varrho_{j,3}^{(0)} \in \mathbb{R}. \tag{3.14}$$

Next suppose that Γ is a lower triangular matrix defined as

$$\Gamma = \Gamma_N(k_1) = (\gamma_{s,l}(k_1))_{N\times N}, \quad \gamma_{s,l}(k_1) = \begin{cases} \frac{1}{(s-l)!}\partial_{k_1}^{s-l}\gamma_1, & s \geq l, \\ 0, & s < l, \end{cases} \tag{3.15}$$

where γ_1 is defined by (3.13b). In this case, the generic entry vector ψ can be taken as

$$\psi = \sum_{s=1}^{3} \mathcal{A}_s \mathcal{Q}_s(k_1), \tag{3.16a}$$

where

$$\begin{aligned} &\mathcal{Q}_s(k_1) = (Q_{s,0}(k_1), Q_{s,1}(k_1), \cdots, Q_{s,N-1}(k_1))^T, \\ &Q_{s,j}(k_1) = \tfrac{1}{j!}\partial_{k_1}^{j}\left[\varrho_{1,s}^{(0)}(\delta - \omega^s k_1)^l (a - \omega^s k_1)^n (b - \omega^s k_1)^m\right], \end{aligned} \tag{3.16b}$$

and $\{\mathcal{A}_s\}$ are arbitrary Nth-order lower triangular Toeplitz matrices defined by

$$\mathcal{A}_s = \begin{pmatrix} a_{s,0} & 0 & 0 & \cdots & 0 & 0 \\ a_{s,1} & a_{s,0} & 0 & \cdots & 0 & 0 \\ a_{s,2} & a_{s,1} & a_{s,0} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{s,N-1} & a_{s,N-2} & a_{s,N-3} & \cdots & a_{s,1} & a_{s,0} \end{pmatrix}_{N\times N}, \quad a_{s,j} \in \mathbb{C}.$$

To get a real solution, in addition to the condition (3.14) we also need $\mathcal{A}_2 = \mathcal{A}_1^*$ and $\mathcal{A}_3$ being real. Reduction to the solution of the lattice BSQ (1.1) are obtained by taking $\delta = 0$ in (3.16b).

Note that in this case there is only one k_1, so solutions corresponding to such a $\Gamma_N(k_1)$ are kind of limit solutions. In fact, let us start from the Casoratians with ψ_j defined in (3.1) and set $\varrho_{j,s}^{(0)} = \varrho_{1,s}^{(0)}$ for $j = 2, 3, \cdots, N$. First we replace the Casoratians f, g, h, s in the bilinear equations (3.9) by $f/K, g/K, h/K, s/K$ with $K = \prod_{j=2}^{N} \frac{(k_j - k_1)^{j-1}}{(j-1)!}$. Then, using L'Hospital rule we can take the limits $k_j \to k_1$ from $j = 2$ to $j = N$ step by step, and finally reach the bilinear equations (3.9) solved by Casoratians f, g, h, s with entry vector $\psi = \sum_{s=1}^{3} \mathcal{Q}_s(k_1)$ where $\mathcal{Q}_s(k_1)$ is defined as (3.16b). The Toeplitz matrix $\mathcal{A}_s$ in (3.16a) can be obtained by a suitable redefinition of $\varrho_{1,s}^{(0)}$ as a function sufficiently differentiable w.r.t k_1 and $\varrho_{j,s}^{(0)} = \varrho_{1,s}^{(0)}(k_j)$ for $j = 2, 3, \cdots, N$.

From the above solution we can also derive rational solutions, by taking a particular choice of parameters in the limit $k_1 \to 0$.

4 Conclusions

In this paper we have bilinearized the lattice Boussinesq equation (1.1) and constructed its multi-soliton solutions in terms of Casoratians. The method, which relies heavily on 3D-consistency, is similar to the one used in [7, 8], with the main difference that the analogue of the exponential factor now contains cubic roots of unity, see (2.14), allowing two different ρ-terms in the bilinear construction. Cubic roots of unity also enter in the continuum case, because the continuum Boussinesq equation is obtained as a three-reduction from the KP equation.

After this work was completed we were informed about reference [12] where the authors bilinearized the lattice BSQ in its 9-point 1-component form using singularity confinement and then constructed its solutions in Casoratian form. Their final result is similar to the present except that it only contained two terms in the Casoratian entries (cf. (3.1)).

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A Proof of Casoratian solutions

Usually in Casoratian proof the bilinear equation is reduced to a Laplace expansion of a $2N\times 2N$ determinant that can be seen to be identically zero. The expansion is described as:

Lemma 1. *[9] Suppose that* $\mathbf{B}$ *is an* $N\times(N-2)$ *matrix and* $\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d}$ *are* N*'th-order column vectors, then*

$$|\mathbf{B},\mathbf{a},\mathbf{b}||\mathbf{B},\mathbf{c},\mathbf{d}|-|\mathbf{B},\mathbf{a},\mathbf{c}||\mathbf{B},\mathbf{b},\mathbf{d}|+|\mathbf{B},\mathbf{a},\mathbf{d}||\mathbf{B},\mathbf{b},\mathbf{c}|=0. \tag{A.1}$$

This Lemma is also used to generate Casoratian equalities by means of which one can simplify Casoratian proofs.

A.1 Formulae for Casoratians

In order to use the above Lemma we need various formulae for the shifts of the Casoratians f,g,h,s of (3.5) with entries (3.1), as given below. These formulae can be derived using (3.8a),(3.8b) in the same way as in [8]. For convenience we introduce an up-shift operator E^ν by

$$E^1\psi\equiv\widetilde{\psi},\quad E^2\psi\equiv\widehat{\psi},\quad E^3\psi\equiv\overline{\psi}.$$

Down shifts are denoted by E_ν, $\nu=1,2,3$, respectively.

The basic shift formulae are

$$-(\alpha_\mu-\delta)^{N-2}E_\mu f=|\widehat{N-2},E_\mu\psi(N-2)|, \tag{A.2a}$$

$$-(\alpha_\mu-\delta)^{N-2}E_\mu[g+(\alpha_\mu-\delta)f]=|\widehat{N-3},N-1,E_\mu\psi(N-2)|, \tag{A.2b}$$

$$-(\alpha_\mu-\delta)^{N-2}E_\mu[h+(\alpha_\mu-\delta)g]=|\widehat{N-3},N,E_\mu\psi(N-2)|, \tag{A.2c}$$

$$(a-b)(a-\delta)^{N-2}(b-\delta)^{N-2}\underset{\hat{\sim}}{f}=|\widehat{N-3},\underset{\hat{}}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|, \tag{A.2d}$$

$$\begin{aligned}&(a-b)(a-\delta)^{N-2}(b-\delta)^{N-2}\Big[\underset{\hat{\sim}}{g}+(a+b-2\delta)\underset{\hat{\sim}}{f}\Big]\\&\quad=-(a-\delta)^{N-2}\underset{\sim}{f}+(b-\delta)^{N-2}\underset{\hat{}}{f}+|\widehat{N-4},N-2,\underset{\hat{}}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|,\end{aligned} \tag{A.2e}$$

$$\begin{aligned}&(a-b)(a-\delta)^{N-2}(b-\delta)^{N-2}\Big[\underset{\hat{\sim}}{s}+(a+b-2\delta)\underset{\hat{\sim}}{g}+[(a-\delta)^2+(a-\delta)(b-\delta)+(b-\delta)^2]\underset{\hat{\sim}}{f}\Big]\\&\quad=-(b-\delta)(a-\delta)^{N-2}\underset{\sim}{f}+(a-\delta)(b-\delta)^{N-2}\underset{\hat{}}{f}+|\widehat{N-5},N-3,N-2,\underset{\hat{}}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|.\end{aligned} \tag{A.2f}$$

where $\mu=1,2$, and $\alpha_1=a,\alpha_2=b$.

In order to apply Lemma 1 in the proof of the main result we need some further equalities (also derived using Lemma 1):

$$\begin{aligned}f|\widehat{N-5},N-3,&N-2,\underset{\hat{}}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\&=-(b-\delta)^{N-2}\underset{\hat{}}{f}|\widehat{N-5},N-3,N-2,N-1,\underset{\sim}{\psi}(N-2)|\\&\quad+(a-\delta)^{N-2}\underset{\sim}{f}|\widehat{N-5},N-3,N-2,N-1,\underset{\hat{}}{\psi}(N-2)|,\end{aligned} \tag{A.3a}$$

$$\begin{aligned}f|\widehat{N-4},N-2,&\underset{\hat{}}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\&=-(b-\delta)^{N-2}\underset{\hat{}}{f}|\widehat{N-4},N-2,N-1,\underset{\sim}{\psi}(N-2)|\\&\quad+(a-\delta)^{N-2}\underset{\sim}{f}|\widehat{N-4},N-2,N-1,\underset{\hat{}}{\psi}(N-2)|\\&=(b-\delta)^{N-2}\underset{\hat{}}{f}\Big[(a-\delta)^{N-2}\underset{\sim}{s}+(a-\delta)^{N-1}\underset{\sim}{g}+(a-\delta)^{N}\underset{\sim}{f}-f\Big]\\&\quad-(a-\delta)^{N-2}\underset{\sim}{f}\Big[(b-\delta)^{N-2}\underset{\hat{}}{s}+(b-\delta)^{N-1}\underset{\hat{}}{g}+(b-\delta)^{N}\underset{\sim}{f}-f\Big],\end{aligned} \tag{A.3b}$$

$$\begin{aligned}g|\widehat{N-4},N-2,&\underset{\hat{}}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\&=-(b-\delta)^{N-2}\underset{\hat{}}{f}|\widehat{N-4},N-2,N,\underset{\sim}{\psi}(N-2)|\\&\quad+(a-\delta)^{N-2}\underset{\sim}{f}|\widehat{N-4},N-2,N,\underset{\hat{}}{\psi}(N-2)|,\end{aligned} \tag{A.3c}$$

$$(a-b)\underset{\hat{\sim}}{f}g=\underset{\hat{}}{f}[\underset{\sim}{h}+(a-\delta)\underset{\sim}{g}]-\underset{\sim}{f}[\underset{\hat{}}{h}+(b-\delta)\underset{\hat{}}{g}], \tag{A.3d}$$

$$\begin{aligned}(a-b)(a-\delta)^{N-2}(b-\delta)^{N-2}\underset{\hat{\sim}}{f}h=&(a-\delta)^{N-2}\underset{\sim}{f}|\widehat{N-3},N+1,\underset{\hat{}}{\psi}(N-2)|\\&-(b-\delta)^{N-2}\underset{\hat{}}{f}|\widehat{N-3},N+1,\underset{\sim}{\psi}(N-2)|.\end{aligned} \tag{A.3e}$$

For the next identity we also need the following Lemma:

Lemma 2. *[9]*

$$\sum_{j=1}^{N}|\mathbf{a}_1,\cdots,\mathbf{a}_{j-1},\,\mathbf{b}\mathbf{a}_j,\,\mathbf{a}_{j+1},\cdots,\mathbf{a}_N|=\Bigg(\sum_{j=1}^{N}b_j\Bigg)|\mathbf{a}_1,\cdots,\mathbf{a}_N|, \tag{A.4}$$

where $\mathbf{a}_j = (a_{1j}, \cdots, a_{Nj})^T$ *and* $\mathbf{b} = (b_1, \cdots, b_N)^T$ *are* N*'th-order column vectors, and* $\mathbf{ba}_j$ *stands for* $(b_1 a_{1j}, \cdots, b_N a_{Nj})^T$.

Then, noting that

$$(\delta - \omega k_j)^3 - 3\delta(\delta - \omega k_j)^2 + 3\delta^2(\delta - \omega k_j) \equiv (\delta - k_j)^3 - 3\delta(\delta - k_j)^2 + 3\delta^2(\delta - k_j) = \gamma_j,$$

we get, using Lemma 2 and the following identity

$$\Big[\Big(\sum_{j=1}^{N} \gamma_j\Big) \underset{\sim}{f}\Big] \underset{\wedge}{f} = \Big[\Big(\sum_{j=1}^{N} \gamma_j\Big) \underset{\wedge}{f}\Big] \underset{\sim}{f}, \tag{A.5}$$

the explicit result

$$\begin{aligned}
&(a-\delta)^{N-2} \underset{\sim}{f} \Big[|\widehat{N-5}, N-3, N-2, N-1, \underset{\wedge}{\psi}(N-2)| \\
&\qquad - |\widehat{N-4}, N-2, N, \underset{\wedge}{\psi}(N-2)| + |\widehat{N-3}, N+1, \underset{\wedge}{\psi}(N-2)| \\
&\qquad + (b-\delta)^{N+1} \underset{\wedge}{f} + g - (b-\delta) f \\
&\qquad - 3\delta(b-\delta)^{N-2}(\underset{\wedge}{s} - \underset{\wedge}{h}) - 3\delta^2 (b-\delta)^{N-2} \underset{\wedge}{g} \Big] \\
&- (b-\delta)^{N-2} \underset{\wedge}{f} \Big[|\widehat{N-5}, N-3, N-2, N-1, \underset{\sim}{\psi}(N-2)| \\
&\qquad - |\widehat{N-4}, N-2, N, \underset{\sim}{\psi}(N-2)| + |\widehat{N-3}, N+1, \underset{\sim}{\psi}(N-2)| \\
&\qquad + (a-\delta)^{N+1} \underset{\sim}{f} + g - (a-\delta) f \\
&\qquad - 3\delta(a-\delta)^{N-2}(\underset{\sim}{s} - \underset{\sim}{h}) - 3\delta^2 (a-\delta)^{N-2} \underset{\sim}{g} \Big] \\
= {}& 0.
\end{aligned} \tag{A.6}$$

A.2 Proof for bilinear equations (3.9)

With these Casoratian formulae derived in Appendix A.1, we can prove bilinear equations (3.9).

Proof for (3.9a): We consider the down-tilde-shifted $\mathcal{B}_1^{\delta}$, i.e.,

$$f[\underset{\sim}{h} + (a-\delta)\underset{\sim}{g}] - g[\underset{\sim}{g} + (a-\delta)\underset{\sim}{f}] + \underset{\sim}{f} s = 0. \tag{A.7}$$

Using (A.2c), (A.2b) and (A.2a) we have

$$\begin{aligned}
& -(a-\delta)^{N-2}[f[\underset{\sim}{h} + (a-\delta)\underset{\sim}{g}] - g[\underset{\sim}{g} + (a-\delta)\underset{\sim}{f}] + \underset{\sim}{f} s] \\
= \; & |\widehat{N-1}||\widehat{N-3}, N, \underset{\sim}{\psi}(N-2)| - |\widehat{N-2}, N||\widehat{N-3}, N-1, \underset{\sim}{\psi}(N-2)| \\
& + |\widehat{N-2}, \underset{\sim}{\psi}(N-2)||\widehat{N-3}, N-1, N|,
\end{aligned}$$

which is zero in the light of Lemma 1 by taking

$$\mathbf{B} = (\widehat{N-3}), \quad \mathbf{a} = \psi(N-2), \quad \mathbf{b} = \psi(N-1), \quad \mathbf{c} = \psi(N), \quad \mathbf{d} = \underset{\sim}{\psi}(N-2).$$

(3.9b) can be proved similarly.

Proof for (3.9c): We prove it in its down-tilde-hat-shifted version:

$$\underset{\wedge}{f}[\underset{\sim}{g}+(a-\delta)\underset{\sim}{f}]-\underset{\sim}{f}[\underset{\wedge}{g}+(b-\delta)\underset{\wedge}{f}]-(a-b)f\underset{\hat{\sim}}{f}=0. \tag{A.8}$$

This can be verified by using first (A.2a), (A.2b), (A.2d) and then Lemma 1 with $\mathbf{B}=(\widehat{N-3})$, $\mathbf{a}=\psi(N-2)$, $\mathbf{b}=\psi(N-1)$, $\mathbf{c}=\underset{\wedge}{\psi}(N-2)$ and $\mathbf{d}=\underset{\sim}{\psi}(N-2)$.

Proof for (3.9d): We first rewrite (3.9d) in the following form

$$f\left[\underset{\hat{\sim}}{s}+(a+b+\delta)\underset{\hat{\sim}}{g}+(a^2+ab+b^2)\underset{\hat{\sim}}{f}\right]-g\left[\underset{\hat{\sim}}{g}+(a+b+\delta)\underset{\hat{\sim}}{f}\right]-(a^2+ab+b^2)\underset{\sim}{f}\underset{\wedge}{f}+\underset{\hat{\sim}}{f}h=0. \tag{A.9}$$

Then using (A.2e) and (A.2f) we have

$$\begin{aligned}
& (a-b)(a-\delta)^{N-2}(b-\delta)^{N-2}\times l.h.s.(\mathrm{A}.9)\\
= \; & f|\widehat{N-5},N-3,N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\
& +(3\delta f-g)|\widehat{N-4},N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|\\
& +(a-b)(a-\delta)^{N-2}(b-\delta)^{N-2}\underset{\hat{\sim}}{f}(3\delta^2 f-3\delta g+h)\\
& +(a-\delta)^{N-2}\underset{\sim}{f}[-(b+2\delta)f+g+b^3(b-\delta)^{N-2}\underset{\wedge}{f}]\\
& -(b-\delta)^{N-2}\underset{\wedge}{f}[-(a+2\delta)f+g+a^3(a-\delta)^{N-2}\underset{\sim}{f}].
\end{aligned}$$

Next, we replace $f|\widehat{N-5},N-3,N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|$ by (A.3a), $f|\widehat{N-4},N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|$ by (A.3b), $g|\widehat{N-4},N-2,\underset{\wedge}{\psi}(N-2),\underset{\sim}{\psi}(N-2)|$ by (A.3c), $\underset{\hat{\sim}}{f}f$ by (A.8), $\underset{\hat{\sim}}{f}g$ by (A.3d) and $\underset{\hat{\sim}}{f}h$ by (A.3e). Then we find the remaining is nothing but the l.h.s.of (A.6), which is zero. $\square$

A.3 Proof in the case of a generic Γ matrix

In fact, if ψ satisfies (3.8a) and (3.8b), then we can get formulae (A.2) and we only need a modification for the proof of the identity (A.6). In the proof we need the following

Lemma 3. *[10] (see also [11]) Suppose that Ξ is an $N\times N$ matrix with column vector set $\{\Xi_j\}$; Ω is an $N\times N$ operator matrix with column vector set $\{\Omega_j\}$ and each entry $\Omega_{j,s}$ being an operator. Then we have*

$$\sum_{j=1}^{N}|\Omega_j*\Xi|=\sum_{j=1}^{N}|(\Omega^T)_j*\Xi^T|, \tag{A.10}$$

where for any N'th-order column vectors A_j and B_j we define

$$A_j\circ B_j=(A_{1,j}B_{1,j},\ A_{2,j}B_{2,j},\cdots,A_{N,j}B_{N,j})^T$$

and

$$|A_j*\Xi|=|\Xi_1,\cdots,\Xi_{j-1},\ A_j\circ\Xi_j,\ \Xi_{j+1},\cdots,\Xi_N|$$

In the light of the above Lemma taking $\Omega_{j,s}\equiv(E^3)^3-3\delta(E^3)^2+3\delta^2E^3$ and $\Xi=\underset{\sim}{f}$ or $\underset{\wedge}{f}$ we have

$$(\mathrm{Tr}(\Gamma)\underset{\sim}{f})\underset{\wedge}{f}=(\mathrm{Tr}(\Gamma)\underset{\wedge}{f})\underset{\sim}{f} \tag{A.11}$$

instead of (A.5). This equality together with formulae (A.2) allows us to get the identity (A.6). $\square$